Lecture 3: Stochastic Discount Factor

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Stochastic Discount Factor (SDF)

- A stochastic discount factor is a stochastic process $\{m_{t+s}\}_{s=1}^{\infty}$ such that for any security with payoff x_{t+1} at time $t+1$ the price of that security at time t is

$$P_t = E_t\left[m_{t+1}x_{t+1}\right]$$

- Or

$$1 = E_t\left[m_{t+1}R_{t+1}\right]$$

where

$$R_{t+1} = \frac{x_{t+1}}{P_t}$$

- In the representative consumer model

$$m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}$$

Overview

- Heterogeneous agents with complete vs incomplete markets
- Non Arbitrage Opportunities (**NAO**)
- Hansen-Jagannathan volatility bounds (JPE, 1991).

Basic properties of SDF

- Risk free rate

$$E_t\left(m_{t+1}\right) = \frac{1}{R^f_{t+1}}$$

- In the representative consumer model with power utility function

$$m_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$$

$$\log m_{t+1} = \log\beta - \gamma\Delta\log c_{t+1}$$

- Compensation for risk

$$E_t R_{t+1} - R^f_{t+1} = -\frac{\sigma_t\left(m_{t+1}\right)}{E_t\left(m_{t+1}\right)} corr(m_{t+1}, R_{t+1})\sigma_t\left(R_{t+1}\right)$$

where $\frac{\sigma_t(m_{t+1})}{E_t(m_{t+1})}$ is known as the market price of risk

Constructing SDF under complete markets

- One period world
- S possible states of nature with probability π_s, $s = 1, ..., S$
- Let q_s be the **state-contingent price**, i.e. the price of an **A**rrow-**D**ebreu **security** that payoffs one unit in state s and zero in the remaining states.
- Definition: The **market is complete** if all A-D securities are available.
- Otherwise is an **incomplete market**. The number of A-D securities is equal to S.

Constructing SDF under complete markets

- The price of a security with payoff $\{d_s\}_{s=1}^{S}$ is

$$P(d) = \sum_{s=1}^{S} q_s d_s = \sum_{s=1}^{S} \pi_s \frac{q_s}{\pi_s} d_s = E\left(\frac{q}{\pi} d\right)$$

It follows that SDF is

$$m_s = \frac{q_s}{\pi_s}$$

and we can write

$$P(d) = E(md)$$

Incomplete markets

- One period economy with I agents
- There is uncertainty about the state s
- Trade occurs before uncertainty is resolved.
- Securities: L securities with prices q_l and payoffs $d_{l,s}$ in state s.

Agent i's problem

- Agent i chooses $\theta_{i,l}$ to maximize

$$E_t u^i(c_{i,s}) = \sum_{s=1}^{S} u^i(c_{i,s})\pi_s$$

subject to the budget constraint

$$\sum_{l=1}^{L} q_l \theta_{i,l} \leq w_i$$

where

$$c_{i,s} = y_{i,s} + \sum_{l=1}^{L} d_{l,s}\theta_{i,l}, \text{ for } s = 1, ..., S$$

where $y_{i,s}$ is the endowment in state s, $\theta_{i,l}$ is the number of shares in security l purchased by agent i and w_i is initial wealth of agent i

Agent i's problem

- The Lagrangian is

$$\begin{aligned}\mathcal{L} &= E_t u^i \left(y_{i,s} + \sum_{l=1}^{L} d_{l,s}\theta_{i,l} \right) \\ &+\lambda_i \left(w_i - \sum_{l=1}^{L} q_l \theta_{i,l} \right)\end{aligned}$$

or

$$\begin{aligned}\mathcal{L} &= \pi_1 u^i \left(y_{i,1} + \sum_{l=1}^{L} d_{l,1}\theta_{i,l} \right) + \pi_2 u^i \left(y_{i,2} + \sum_{l=1}^{L} d_{l,2}\theta_{i,l} \right) + \ldots \\ &+\pi_S u^i \left(y_{i,S} + \sum_{l=1}^{L} d_{l,S}\theta_{i,l} \right) + \lambda_i \left(w_i - \sum_{l=1}^{L} q_l \theta_{i,l} \right)\end{aligned}$$

where λ_i is agent i's Lagrange multiplier on the wealth constraint.

Agent i's problem

- The Lagrangian is

$$\begin{aligned}\mathcal{L} &= \sum_{s=1}^{S} \pi_s u^i \left(y_{i,s} + \sum_{l=1}^{L} d_{l,s}\theta_{i,l} \right) \\ &\quad + \lambda_i \left(w_i - \sum_{l=1}^{L} q_l \theta_{i,l} \right)\end{aligned}$$

- First order conditions:

$$\sum_{s=1}^{S} \frac{\partial u^i}{\partial c_{i,s}} d_{l,s}\pi_s = q_l \lambda_i, \text{ for } l = 1, ..., L$$

Agent i's problem

- A price vector $q = \{q_l\}$ and shares $\{\theta_{i,l}\}$ are such that each agent maximizes utility and markets clear. Total supply of shares can be without loss of generality normalized to 1.

$$\sum_{i=1}^{I} \theta_{i,l} = 1, \text{ for } l = 1, .., L$$

- SDF: Define

$$m_{i,s} = \frac{\frac{\partial u^i}{\partial c_{i,s}}}{\lambda_i}$$

The Euler equation is

$$q_l = E_t m_{i,s} d_{l,s} = \sum_{s=1}^{S} m_{i,s} d_{l,s} \pi_s$$

We can use the marginal rate of substitution of any consumer who is unconstrained (on the choice of securities) as an SDF.

- Agents may be constrained in each state but as long as there are no arbitrage opportunities there exists an SDF.

Incomplete markets: no arbitrage pricing

- **Definition**: There is **NAO** when does not exist a portfolio with a zero (or negative) price that gives nonnegative payoffs in all states of the world and a strictly positive payoff in at leat one state of the world.
- Formally: The system of securities characterized by $\{q_l\}$ and $\{d_{l,s}\}$ is arbitrage free if there is no vector of portfolio choices $\{\theta_l\}$ such that both

$$\sum_{l=1}^{L} q_l \theta_l \leq 0$$

and for all $s = 1, ..., S$

$$\sum_{l=1}^{L} d_{l,s} \theta_l \geq 0$$

and > 0 for some states

NAO and SDF

- **Assumptions:** no short-selling constraints, no bid ask spreads, no transaction costs, no taxes
- **Proposition**: There are NAO (even if markets are incomplete) if and only if there is some positive SDF
- **Proof:** If there is a positive SDF,

$$m_{t+1} = m_s > 0 \text{ for each of the } s \text{ realizations}$$

by multiplying the payoff of portfolio θ in state s

$$\sum_{l=1}^{L} d_{l,s}\theta_l \geq 0, \text{ for all } s = 1, ..., S \tag{1}$$

by $\pi_s m_s$ and adding up across s get

$$\sum_{s=1}^{S} \pi_s m_s \sum_{l=1}^{L} d_{l,s}\theta_l \geq 0$$

NAO and SDF

- $$\sum_{l=1}^{L} \theta_l \sum_{s=1}^{S} \pi_s m_s d_{l,s} = \sum_{l=1}^{L} \theta_l E_t \left(m_{t+1} d_l \right) = \sum_{l=1}^{L} \theta_l q_l \geq 0 \qquad (2)$$

  Hence, if any of the s inequalities (1) is strictly positive then (2) will be strictly positive, which implies that there are NAO.
- The other part of the proof, that if the are NAO a discount factor must exist, is more demanding. The discount factor can be constructed by using the hyperplane separating theorem. (see for instance Cochrane's *Asset Pricing*)

Implications of no arbitrage pricing

- If prices $\{q_l\}$ result from a competitive equilibrium then there will be no arbitrage – otherwise the demand of that security would be infinite.
- More generally we can use no arbitrage theory to place restrictions on prices. Example: if asset x pays more dividends than asset y in all states then the price of asset x must be (weakly) greater than the price of asset y.
- Assets that are redundant, i.e. can be replicated or "spanned" by other assets can be priced even if markets are incomplete.

Example

- $S = 5$, $L = 3$

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 3 & 2 \end{bmatrix}$$

- The third asset payoff is equal to the payoff of the first plus the payoff of the second. Thus, its price must be equal to the sum of the price of the first and second.
- Under NAO, options can be priced as linear combinations of equity and debt payouts.

European Call Options

- Options can generate contingent claims

$$C_T(K) = \max(P_T - K, 0)$$

$C_T(K)$ = value at expiration (T) of a call (on a stock) with strike price K.
P_t = stock price today
P_T = stock price at expiration.

- Assume to simplify that there are S states and $P_T = \{1, 2, 3, ..., S\}$
- An asset with payoff $(1, 0, 0, ..., 0)$, can be generated with $[C_T(0) - C_T(1)] - [C_T(1) - C_T(2)] = (1, 1, 1, ..., 1) - (0, 1, 1, ..., 1)$
- What about $(0, 0, 0, ..., 1)$? Answer: $C_T(S-1)$.
- What about $(0, 0, 0, ..., 1, 0)$? Answer: $C_T(S-2) - 2C_T(S-1)$.

LOOP and SDF

- The **L**aw **O**f **O**ne **P**rice says that a given payoff has a unique price.
- LOOP holds iff there is a unique stochastic discount factor.
  - LOOP implies prices are linear functions of payoffs: $P(x+y) = P(x) + P(y)$
  - This implies that the price of a security with payoff $\{x\}$ is

$$P(x) = \sum_{s=1}^{S} q_s x_s$$

  where q_s is the state-contingent price

- Define

$$m_s = \frac{q_s}{\pi_s}$$

  then

$$P(x) = \sum_{s=1}^{S} \pi_s m_s x_s = E(mx)$$

  so that m_s is an SDF.

LOOP and SDF

- if there is SDF m_s then the price of securities $\{x\}$ and $\{y\}$ are

$$P(x+y) = E\left(m\left(x+y\right)\right) = E\left(mx\right) + E\left(my\right) = P(x) + P(y)$$

NAO and SDF under complete markets

- **Proposition:** There is a **unique** discount factor if and only if markets are complete and there are NAO.
- If markets are complete and there are NAO then contingent claims prices $\{q_s\}$ are unique.
  - Otherwise there would be two prices for the two different portfolios that generate the same payoff. This would violate NAO. Investors would sell the most expensive and buy the less expensive.
- **Proof**: If markets are complete then

$$q_s = E_t \{m_{t+1} e^s\}$$

where e^s is the Arrow-Debreu security of state s

- It follows that SDF on entry s is

$$m_s = \frac{q_s}{\pi_s}$$

Thus, the discount factor is unique.

NAO and SDF under complete markets

- The converse: If there is a unique discount factor then markets are complete.
- Suppose not, that there is a unique discount factor m and markets are incomplete. In that case there will be a state j that is non-tradable (i.e. no AD security exists for that state).
- Define another SDF m^*

$$\begin{aligned} m_s^* &= m_s \text{ for all } s \neq j \\ m_j^* &= m_j + 1 \end{aligned}$$

  The SDF m^* will price all tradable securities hence m is not unique.
- Exercise: Show that if m^* and m are two discount factors then $w_1 m^* + w_2 m$ is a discount factor too, with $w_1 + w_2 = 1$

Risk neutral probabilities

- The price of a security with payoffs $\{x\}$ is
  $P(x) = \sum_{s=1}^{S} \pi_s m_s x_s = E(mx)$
- If agents were risk neutral then $m = \beta$ and the value of the security would be its discounted expected payoffs:

$$P(x) = \beta E(x)$$

- **Probabilities** can be redefined so that the current price of a security is a "special" expectation of its payoffs

- **Risk neutral probabilities**

$$p_s = \frac{\pi_s m_s}{\sum_{s=1}^{S} \pi_s m_s} = \frac{\pi_s m_s}{E_t \left\{ m_{t+1} \right\}} = R^f \pi_s m_s, \text{ for } s = 1, 2, ..., S$$

Let $B_{1,t}$ be the price of a security that pays one unit in all states of nature

$$B_{1,t} = E_t \left\{ m_{t+1} \right\} = \sum_{s=1}^{S} \pi_s m_s = \sum_{s=1}^{S} q_s$$

and $R^f = \frac{1}{B_{1,t}}$

Risk neutral probabilities

- Rate of return of security l ($l = 1, ..., L$) can be rewritten as

$$1 = E_t \left\{ m_{t+1} R_l \right\} = \sum_{s=1}^{S} \pi_s m_s R_{l,s}$$

- If we divide this equation by $B_{1,t}$

$$\frac{1}{B_{1,t}} = \sum_{s=1}^{S} \frac{\pi_s m_s}{E_t \left\{ m_{t+1} \right\}} R_{l,s} = \sum_{s=1}^{S} p_s R_{l,s}$$

Risk neutral probabilities

- We can write the **risk neutral valuation formulas** as

$$R^f = \frac{1}{B_{1,t}} = \sum_{s=1}^{S} p_s R_{l,s}, \text{ for all assets } l$$

- Thus: Using probabilities p, the securities l ($l = 1, ..., L$) can be valued as if agents were risk neutral

$$R^f = E_t^p\left(R_l\right), \text{ for all assets } l$$

where the p in E_t^p denotes that the expectation is taken with respect to the artificial probability p.

Risk neutral probabilities

- A security with payoffs $\{x\}$ has a price $P(x)$

$$
\begin{aligned}
P(x) &= \sum_{s=1}^{S} q_s x_s = \sum_{s=1}^{S} \pi_s m_s x_s \\
&= B_{1,t} \sum_{s=1}^{S} \frac{\pi_s m_s}{B_{1,t}} x_s = B_{1,t} \sum_{s=1}^{S} p_s x_s = \frac{E_t^p(x)}{R^f}
\end{aligned}
$$

- Can do asset pricing as if agents are all risk neutral, but with probabilities p instead of the true probabilities π.
- The probabilities p give greater weight to states with higher relative marginal utility.

Risk sharing

- When markets are complete can use the various period budget constraints to obtain a single intertemporal budget constraint
- The AD securities prices are the prices of consumptions
- Take without loss of generality a 2-period model
- The budget constraints are

$$c + \sum_{s=1}^{S} q_s b_s = y \text{ and } c_s = b_s + y_s \text{ for each } s \in S$$

replacing the b_s gives the intertemporal budget constraint

$$c + \sum_{s=1}^{S} q_s c_s = y + \sum_{s=1}^{S} q_s y_s$$

Risk sharing

- The price of the AD security of state s is equal to the product between the probability of state s and the MRS between consumption today and consumption in state s
- The solution of

$$\max_{\{c,c_s\}} u(c) + \sum_{s=1}^{S} \beta u(c_s)\pi_s$$

s.t.

$$c + \sum_{s=1}^{S} q_s c_s = y + \sum_{s=1}^{S} q_s y_s$$

gives the contingent claim price

$$q_s = \pi_s \frac{\beta u'(c_s)}{u'(c)}$$

Risk sharing

- The marginal rate of substitution (MRS) for any individual investor is equal to $\frac{q_s}{\pi_s}$.
  - But the prices are the same for all investors.
  - Therefore, marginal utility growth should be the same for all investors

$$\frac{\beta u'(c^i_{s,t+1})}{u'(c^i_t)} = \frac{\beta u'(c^j_{s,t+1})}{u'(c^j_t)}, \text{ for } s = 1, ..., S$$

  where i and j refer to different investors. If investors have the same wealth then $c^i_t = c^j_t$.

- With complete markets, all investors share all risks, so when any shock hits, it hits all equally.
  - Idiosyncratic risk does not matter, only aggregate risk matters.
- **Conclusion**: Security markets – state-contingent claims – bring individual consumptions closer together by allowing people to share idiosyncratic risks.

Risk sharing

- What is the relationship between complete markets and Pareto optimality (in a pure endowment economy)?
- Let there be 2 agents i and j.
- The solution that maximizes social planner utility given weights λ_i and λ_j and the available resources must solve the problem:

$$\max_{\left\{c^i, c^j, c^i_s, c^j_s\right\}} \lambda_i u(c^i) + \lambda_j u(c^j) + \beta \sum_s \pi_{k_t} \left(\lambda_i u(c^i_s) + \lambda_j u(c^j_s)\right)$$

s.t.

$$c^i + c^j = y^i_s + y^j_s \text{ and } c^i_s + c^j_s = y^i_s + y^j_s, \text{ for all } s$$

implies

$$\lambda_i u'(c^i) = \lambda_j u'(c^j),$$

$$\lambda_i u'(c^i_s) = \lambda_j u'(c^j_s), \text{ for all states } s$$

Risk sharing

- Again
$$\lambda_i u'(c_s^i) = \lambda_j u'(c_s^j), \text{ for all states } s$$
- If the social planner likes everyone equally, $\lambda_i = \lambda_j$, then agents consume the same in each state of nature
- If $\lambda_i = \frac{\beta}{u'(c_t^i)}$ and $\lambda_j = \frac{\beta}{u'(c_t^j)}$ the equilibrium with complete markets is a Pareto optimum.
- If the aggregate amount of the good is the same in each state of nature and date, i.e. $c_s^i + c_s^j = c_s$, for all s then
$$q_s = \pi_s \frac{\beta u'(c_s)}{u'(c)} = \pi_s \beta$$
and the riskless return is:
$$R = \beta^{-1}$$

Hansen-Jagannathan Bounds

- Consumption based models don't work empirically – equity premium puzzle.
- Instead of just trying a bunch of different utility functions, it is helpful to characterize some properties that m must satisfy.
- HJ bounds – bound on $\{\sigma(m), E(m)$, other moments of $m\}$
- Purpose:
  - Give us a clearer understanding of why certain asset pricing models are rejected by the data.
  - Allow us to compare asset pricing models against one another.

- Consider any risky return R and risk-free return R^f then
  -
$$E(mR) = 1$$
$$E(m)R^f = 1$$
$$E(mR) = E(m)R^f$$

    This implies:

$$\begin{aligned} cov(m, R) &= E(mR) - E(m)E(R) \\ &= E(m)R^f - E(m)E(R) \end{aligned}$$

Hansen-Jagannathan Bounds

- Now use bound $(-1 \leq corr(x,y) \leq 1)$:

$$-\sigma(x)\sigma(y) \leq cov(x,y) \leq \sigma(x)\sigma(y)$$

to obtain

$$-\sigma(m)\sigma(R) \leq E(m)R^f - E(m)E(R) \leq \sigma(m)\sigma(R)$$

Divide the inequality by $E(m)\sigma(R)$ to obtain:

$$-\frac{\sigma(m)}{E(m)} \leq \frac{R^f - E(R)}{\sigma(R)} \leq \frac{\sigma(m)}{E(m)}$$

or

$$-\frac{\sigma(m)}{E(m)} \leq \frac{E(R) - R^f}{\sigma(R)} \leq \frac{\sigma(m)}{E(m)}$$

Comments

- So $\frac{\sigma(m)}{E(m)}$ must be at least as large as $\frac{E(R)-R^f}{\sigma(R)}$.
- $\frac{E(R)-R^f}{\sigma(R)}$ is the **Sharpe ratio** and $\frac{\sigma(m)}{E(m)}$ is the **market price of risk**.
- The Sharpe ratio for the market is $0.06/0.17 = 35\%$.
- This applies to any asset – for some assets the Sharpe ratios can be even higher so the price of risk must be very high.

Price of risk under CRRA

For random-walk consumption:

$$\Delta \ln c_t = \mu_t + \varepsilon_t, \text{ where } \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

Results from log-normal

$$\begin{aligned} E\left(\frac{c_{t+1}}{c_t}\right) &= \exp\left(\mu + \frac{1}{2}\sigma_\varepsilon^2\right) \\ var\left(\frac{c_{t+1}}{c_t}\right) &= \left(e^{\sigma_\varepsilon^2} - 1\right) e^{2\mu + \sigma_\varepsilon^2} \end{aligned}$$

Price of risk under CRRA

These results imply:

$$E(m) = \beta \exp\left[\gamma\left(\mu + \frac{1}{2}\sigma_{\varepsilon}^{2}\right)\right]$$

and

$$\frac{\sigma(m)}{E(m)} = \left(e^{\gamma^{2}\sigma_{\varepsilon}^{2}} - 1\right)^{1/2}$$

where γ is the CRRA coefficient. Quarterly calibration: $\sigma_{\varepsilon} = 0.036/4$
If $\gamma = 1$ price of risk $= 0.01$
If $\gamma = 10$ price of risk $= 0.09$
If $\gamma = 20$ price of risk $= 0.18$
If $\gamma = 35$ price of risk $= 0.33$
If $\gamma = 50$ price of risk $= 0.47$
To match the Sharpe ratio for the market need a $\gamma > 35$

How to construct a discount factor?

- **Example**: Consider the returns of N securities: $\mathbf{R} = \begin{bmatrix} R_1 \\ R_2 \\ ... \\ R_N \end{bmatrix} \in \mathcal{R}^S$,

  each of the R_is is a vector with S entries that correspond to the number of states in the economy
- Guess and verify approach:

$$m^* = \mathbf{1}' E \left(\mathbf{R}\mathbf{R}' \right)^{-1} \mathbf{R}$$

where $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix} \in \mathcal{R}^S$.

How to construct a discount factor?

- Observe that for all returns in $\mathbf{R}$

$$\begin{aligned} E_t\left\{m^*_{t+1}\mathbf{R}'\right\} &= E_t\left\{\mathbf{1}'E_t\left(\mathbf{R}\mathbf{R}'\right)^{-1}\mathbf{R}\mathbf{R}'\right\} \\ &= \left\{\mathbf{1}'E_t\left(\mathbf{R}\mathbf{R}'\right)^{-1}E_t\left(\mathbf{R}\mathbf{R}'\right)\right\} = \mathbf{1}' \end{aligned}$$

How to construct a discount factor?

- This sample discount factor $m^* = \mathbf{1}' E_t \left(\mathbf{RR}' \right)^{-1} \mathbf{R}$ is a weighted average, with weigths $E_t \left(\mathbf{R} \left[\mathbf{R} \right]' \right)^{-1}$, of all returns in the sample.
- This portfolio prices perfectly all returns in the sample.
- Thus, it works very well in the sample.
- However, typically the discount factors that are constructed to work well in the sample do not price so well the returns out of sample.
- For instance the Fama-French 3 Factors discount factor in general perform better than the discount factors of this type out of sample.